

# A FINITE ELEMENT APPROXIMATION OF THE ONE-DIMENSIONAL FRACTIONAL POISSON EQUATION WITH APPLICATIONS TO NUMERICAL CONTROL

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## 1. THE PROBLEMS CONSIDERED

We obtain a finite element (FE) scheme for the numerical approximation of the solution to the following non-local Poisson equation

$$(1.1) \quad \begin{cases} (-d_x^2)^s u = f, & x \in (-L, L) \\ u \equiv 0, & x \in \mathbb{R} \setminus (-L, L). \end{cases}$$

In (1.1),  $f$  is a given function and, for all  $s \in (0, 1)$ ,  $(-d_x^2)^s$  denotes the one-dimensional fractional Laplace operator, which is defined as the following singular integral

$$(1.2) \quad (-d_x^2)^s u(x) = c_{1,s} \text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy.$$

Here,  $c_{1,s}$  is an explicit normalization constant.

The main novelty of this work relies on the fact that, since we are dealing with a one-dimensional problem, each entry of the stiffness matrix can be computed explicitly, without requiring numerical integration, which is instead needed for the multidimensional case (see [1]). This has the great advantage of reducing the computational cost and allows to be used on more general applications.

A natural example is given by the numerical resolution of the following control problem: given any  $T > 0$ , find a control function  $g \in L^2((-1, 1) \times (0, T))$  such that the corresponding solution to the parabolic problem

$$(1.3) \quad \begin{cases} z_t + (-d_x^2)^s z = g \mathbf{1}_\omega, & (x, t) \in (-1, 1) \times (0, T) \\ z = 0, & (x, t) \in [\mathbb{R} \setminus (-1, 1)] \times (0, T) \\ z(x, 0) = z_0(x), & x \in (-1, 1) \end{cases}$$

satisfies  $z(x, T) = 0$ .

The approach employed for solving this control problem is based on the penalized Hilbert Uniqueness Method ([2]).

## 2. PRELIMINARY RESULTS

**2.1. Variational formulation of the elliptic problem.** Find  $u \in H_0^s(-L, L)$  such that

$$\frac{c_{1,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+2s}} dx dy = \int_{-L}^L f v dx,$$

for all  $v \in H_0^s(-L, L)$ . Here,  $H_0^s(-L, L)$  denotes the space

$$H_0^s(-L, L) := \left\{ u \in H^s(\mathbb{R}) : u = 0 \text{ in } \mathbb{R} \setminus (-L, L) \right\},$$

while  $H^s(\mathbb{R})$  is the classical fractional Sobolev space of order  $s$ . Since the bilinear form  $a$  is continuous and coercive, Lax-Milgram Theorem immediately implies that, if  $f \in H^{-s}(-L, L)$ , the dual space of  $H_0^s(-L, L)$ , then (1.1) admits a unique weak solution  $u \in H_0^s(-L, L)$ .

## 2.2. Parabolic problem.

**Definition 2.1.** System (1.3) is said to be null-controllable at time  $T$  if, for any  $z_0 \in L^2(-1, 1)$ , there exists  $g \in L^2((-1, 1) \times (0, T))$  such that the corresponding solution  $z$  satisfies

$$z(x, T) = 0.$$

**Definition 2.2.** System (1.3) is said to be approximately controllable at time  $T$  if, for any  $z_0, z_T \in L^2(-1, 1)$  and any  $\delta > 0$ , there exists  $g \in L^2((-1, 1) \times (0, T))$  such that the corresponding solution  $z$  satisfies

$$\|z(x, T) - z_T\|_{L^2(-1, 1)} < \delta.$$

There are few results in the literature on the null-controllability of the fractional heat equation, and they all deal with the *spectral* definition of the fractional Laplace operator, which is introduced, e.g., in [3, 4]. In this framework, it is known that the fractional heat equation is null-controllable, provided that  $s > 1/2$ . For  $s \leq 1/2$ , instead, null controllability does not hold, not even for  $T$  large.

However, also for the parabolic equation involving the integral fractional Laplacian (1.2), at least in the one space dimension, these properties are easily achievable. In more detail, we prove the following.

**Proposition 2.1.** For all  $z_0 \in L^2(-1, 1)$  the parabolic problem (1.3) is null-controllable with a control function  $g \in L^2(\omega \times (0, T))$  if and only if  $s > 1/2$ .

Even if for  $s \leq 1/2$  null controllability for (1.3) fails, we still have the following result of approximate controllability.

**Proposition 2.2.** Let  $s \in (0, 1)$ . For all  $z_0 \in L^2(-1, 1)$ , there exists a control function  $g \in L^2(\omega \times (0, T))$  such that the unique solution  $z$  to the parabolic problem (1.3) is approximately controllable.

## 3. DEVELOPMENT OF THE NUMERICAL SCHEME

**3.1. Finite element approximation of the elliptic problem.** Let us introduce a partition of the interval  $(-L, L)$  as follows:

$$-L = x_0 < x_1 < \dots < x_i < x_{i+1} < \dots < x_{N+1} = L,$$

with  $x_{i+1} = x_i + h$ ,  $i = 0, \dots, N$ ,  $h > 0$ . Having defined  $K_i := [x_i, x_{i+1}]$  we consider the discrete space

$$V_h := \left\{ v \in H_0^s(-L, L) \mid v|_{K_i} \in \mathcal{P}^1 \right\},$$

where  $\mathcal{P}^1$  is the space of the continuous and piece-wise linear functions. If we indicate with  $\{\phi_i\}_{i=1}^N$  the classical basis of  $V_h$  in which each  $\phi_i$  is the tent function with  $\text{supp}(\phi_i) = (x_{i-1}, x_{i+1})$  and verifying  $\phi_i(x_j) = \delta_{i,j}$ , at the end we are reduced to solve the linear system

$$\mathcal{A}_h u = F,$$

where the stiffness matrix  $\mathcal{A}_h \in \mathbb{R}^{N \times N}$  has components

$$(3.1) \quad a_{i,j} = \frac{c_{1,s}}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_i(x) - \phi_i(y))(\phi_j(x) - \phi_j(y))}{|x - y|^{1+2s}} dx dy, \quad i, j = 1, \dots, N,$$

while the vector  $F \in \mathbb{R}^N$  is given by  $F = (F_1, \dots, F_N)$  with

$$F_i = \langle f, \phi_i \rangle = \int_{-L}^L f \phi_i dx, \quad i = 1, \dots, N.$$

The construction of the stiffness matrix  $\mathcal{A}_h$  is done in three steps, since its elements can be computed differentiating among three well defined regions: the upper triangle, corresponding to indices  $j \geq i + 2$ , the upper diagonal corresponding to  $j = i + 1$ , and the diagonal corresponding to  $j = i$ . In each of these regions, the intersections among the support of the basis functions are different, thus generating different values of the bilinear form.

After several computations, we obtain the following expressions for the elements of the stiffness matrix  $\mathcal{A}_h$ : for  $s \neq 1/2$

$$a_{i,j} = -h^{1-2s} \begin{cases} \frac{4(k+1)^{3-2s} + 4(k-1)^{3-2s} - 6k^{3-2s} - (k+2)^{3-2s} - (k-2)^{3-2s}}{2s(1-2s)(1-s)(3-2s)}, & k = j-i, k \geq 2 \\ \frac{3^{3-2s} - 2^{5-2s} + 7}{2s(1-2s)(1-s)(3-2s)}, & j = i+1 \\ \frac{2^{3-2s} - 4}{s(1-2s)(1-s)(3-2s)}, & j = i. \end{cases}$$

For  $s = 1/2$ , instead, we have

$$a_{i,j} = \begin{cases} -4(j-i+1)^2 \log(j-i+1) - 4(j-i-1)^2 \log(j-i-1) \\ \quad + 6(j-i)^2 \log(j-i) + (j-i+2)^2 \log(j-i+2) + (j-i-2)^2 \log(j-i-2), & j > i+2 \\ 56 \ln(2) - 36 \ln(3), & j = i+2. \\ 9 \ln 3 - 16 \ln 2, & j = i+1 \\ 8 \ln 2, & j = i. \end{cases}$$

**3.2. Control problem for the fractional heat equation.** We employ the so called penalised Hilbert Uniqueness Method (HUM) that deals with the resolution of the following minimization problem: find

$$(3.2) \quad \varphi_\varepsilon^T = \min_{\varphi \in L^2(-1,1)} J_\varepsilon(\varphi^T)$$

where

$$(3.3) \quad J_\varepsilon(\varphi^T) := \frac{1}{2} \int_0^T \int_\omega |\varphi|^2 dx dt + \frac{\varepsilon}{2} \|\varphi^T\|_{L^2(-1,1)}^2 + \int_\Omega z_0 \varphi(0) dx,$$

where  $\varphi$  is the solution to the adjoint problem

$$\begin{cases} -\varphi_t + (-d_x^2)^s \varphi = 0, & (x, t) \in (-1, 1) \times (0, T) \\ \varphi = 0, & (x, t) \in [\mathbb{R} \setminus (-1, 1)] \times (0, T) \\ \varphi(x, T) = \varphi^T(x), & x \in (-1, 1). \end{cases}$$

Then, the approximate and null controllability properties of the system, for a given initial datum  $z_0$ , can be expressed in terms of the behavior of the penalized HUM approach. In particular, according to [2, Theorem 1.7] we have:

- The equation is approximately controllable at time  $T$  from the initial datum  $z_0$  if and only if

$$\varphi_\varepsilon^T \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

- The equation is null-controllable at time  $T$  from the initial datum  $z_0$  if and only if

$$M_{z_0}^2 := 2 \sup_{\varepsilon > 0} \left( \inf_{L^2(0,T;L^2(\omega))} J_\varepsilon \right) < +\infty.$$

In this case, we have

$$\begin{aligned} \|g\|_{L^2(0,T;L^2(\omega))} &\leq M_{z_0}, \\ \|\varphi_\varepsilon^T\|_{L^2(-L,L)} &\leq M_{z_0} \sqrt{\varepsilon}. \end{aligned}$$

#### 4. NUMERICAL RESULTS

Here, we present the numerical simulations corresponding to our algorithms. We provide a complete discussion of the results obtained.

First of all, in order to test numerically the accuracy of our method for the discretization of the elliptic equation (1.1) we use the following problem

$$(4.1) \quad \begin{cases} (-d_x^2)^s u = 1, & x \in (-L, L) \\ u \equiv 0, & x \in \mathbb{R} \setminus (-L, L). \end{cases}$$

In this particular case, the solution can be expressed as follows

$$(4.2) \quad u(x) = \frac{2^{-2s} \sqrt{\pi}}{\Gamma\left(\frac{1+2s}{2}\right)\Gamma(1+s)} (L^2 - x^2)^s.$$

In Fig. 1 and Fig. 2, we show a comparison for different values of  $s$  between the exact solution (4.2) and the computed numerical approximation. Here we consider  $L = 1$  and  $N = 50$ . One can notice that, when  $s = 0.1$ , the computed solution is to a certain extent different from the exact solution. However, one should be careful with such result and a more precise analysis of the error should be carried.

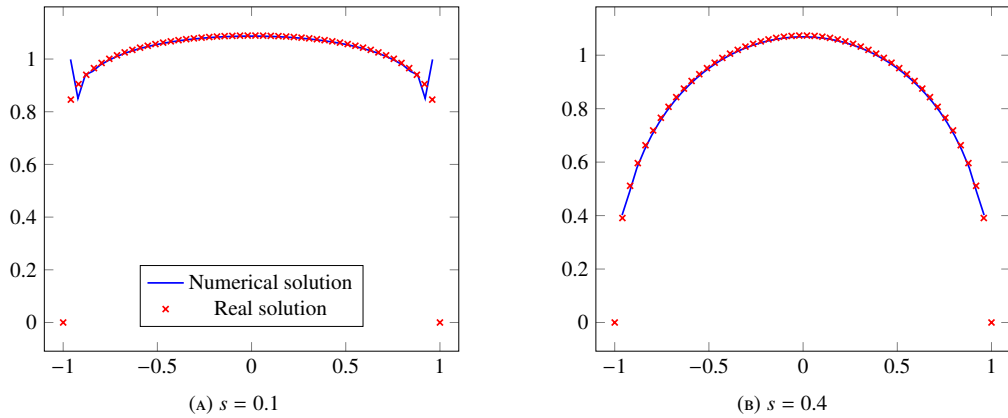


FIGURE 1. Plot for different values of  $s < 1/2$ .

In Fig. 3, we present the computational errors evaluated for different values of  $s$  and  $h$ .

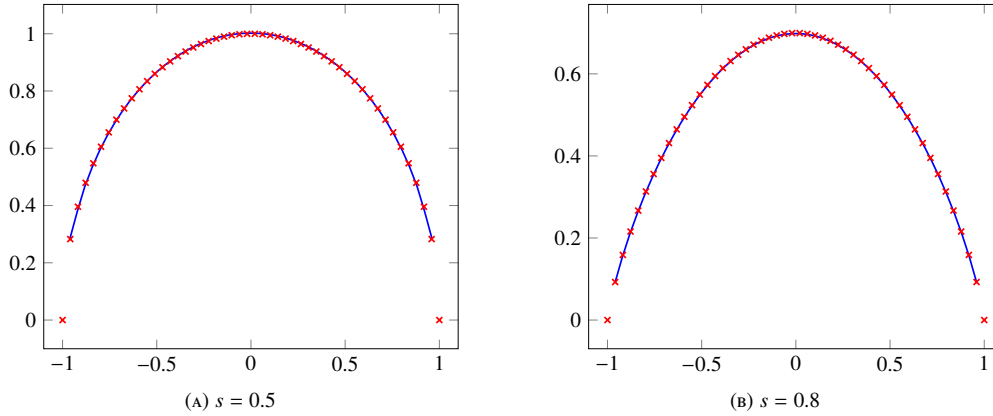
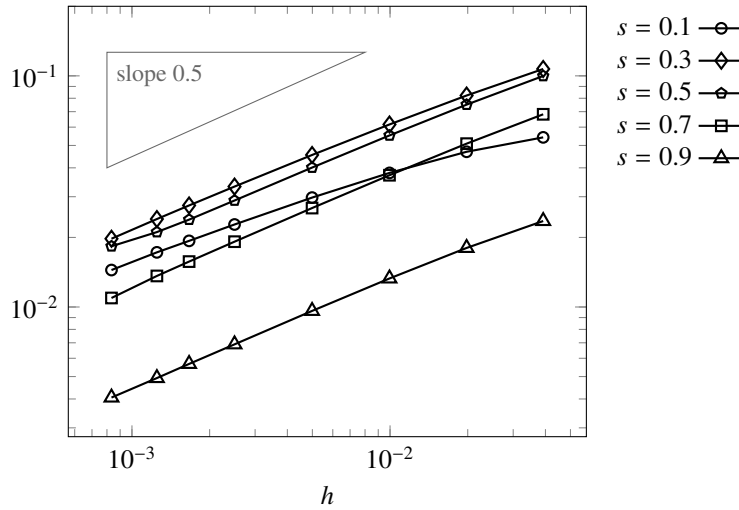

 FIGURE 2. Plot for different values of  $s \geq 1/2$ .


FIGURE 3. Convergence of the error.

The rates of convergence shown are of order (in  $h$ ) of  $1/2$ . This is in accordance with the known theoretical results (see, e.g., [1, Theorem 4.6]). One can see that the convergence rate is maintained also for small values of  $s$ . Since it is well-known that the notion of trace is not defined for the spaces  $H^s(-L, L)$  with  $s \leq 1/2$ , it is somehow natural that we cannot expect a point-wise convergence in this case.

**4.1. Control experiments.** We present some numerical results for the controllability of the fractional heat equation (1.3). We use the finite-element approximation of  $(-d_x^2)^s$  for the space discretization and the implicit Euler scheme in the time variable. We choose the penalization term  $\varepsilon$  as a function of  $h$ . A practical rule ([2]) is to choose  $\phi(h) \sim h^{2p}$  where  $p$  is the order of accuracy in space of the numerical method used for the discretization of the spatial operator involved. In this case, we take  $p = 1/2$ .

In Fig. 4, we observe the time evolution of the uncontrolled solution as well as the controlled solution. Here, we set  $s = 0.8$ ,  $\omega = (-0.3, 0.8)$  and  $T = 0.3$ , and choose  $z_0(x) = \sin(\pi x)$ . The control domain is the highlighted zone on the plane  $(t, x)$ . As expected, we observe that the uncontrolled solution decays with time, but does not reach zero at time  $T$ , while the controlled solution does.

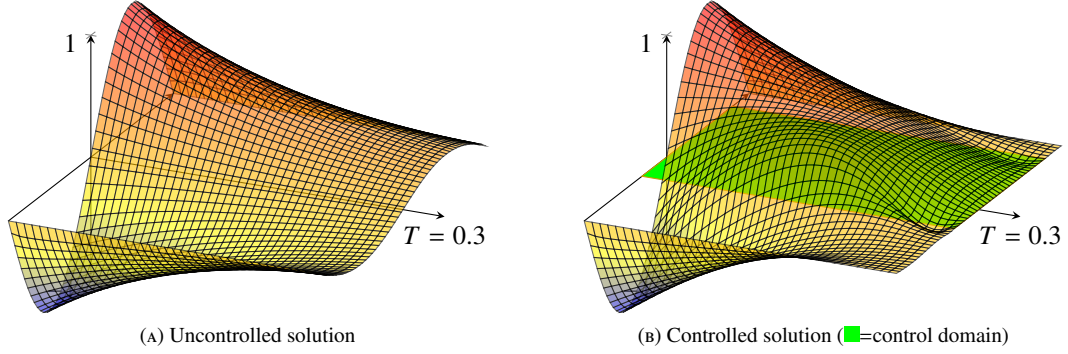


FIGURE 4. Time evolution of system (1.3).

In figure 5, for  $s = 0.8$  we present the computed values of various quantities of interest when the mesh size goes to zero. More precisely, we observe that the control cost and the optimal energy remain bounded as  $h \rightarrow 0$ . On the other hand, we see that

$$(4.3) \quad \|y(T)\|_{L^2(-1,1)} \sim C \sqrt{\phi(h)} = Ch^{1/2}.$$

We know that, for  $s = 0.8$ , system (1.3) is null controllable. This is now confirmed by (4.3), according [2, Theorem 1.7]. In fact, the same experiment can be repeated for different values of  $s > 1/2$ , obtaining the same conclusions.

In Fig. 6, we illustrate the numerical results for the case  $s \leq 1/2$ . We observe that the cost of the control and the optimal energy increase in both cases, while the target  $y(T)$  tends to zero with a slower rate than  $h^{1/2}$ . This seems to confirm that a uniform observability estimate for (1.3) does not hold and that we can only expect to have approximate controllability.

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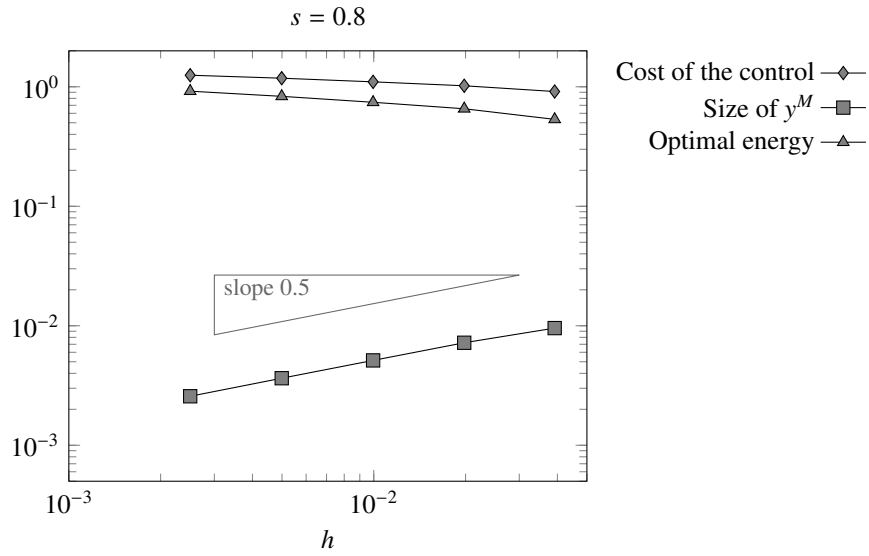


FIGURE 5. Convergence properties of the method for controllability of the fractional heat equation.

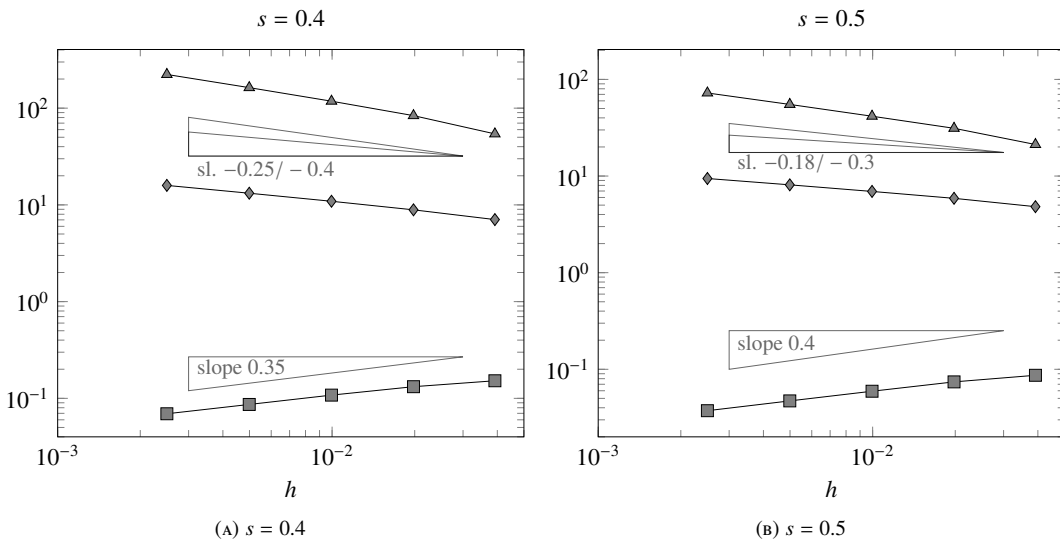


FIGURE 6. Convergence properties of the method for  $s < 1/2$ . Same legend as in Fig. 5